Problem 12

If f is a differentiable function and $g(x) = xf(x)$, use the definition of a derivative to show that $g'(x) = xf'(x) + f(x)$.

Solution

Use the definition of the derivative to determine $g'(x)$.

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)f(x+h) - xf(x)}{h} \\
&= \lim_{h\to 0} \frac{xf(x+h) + hf(x+h) - xf(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{xf(x+h) - xf(x)}{h} + \frac{hf(x+h)}{h}\right] \\
&= \lim_{h\to 0} \frac{xf(x+h) - xf(x)}{h} + \lim_{h\to 0} \frac{hf(x+h)}{h} \\
&= \lim_{h\to 0} \frac{x[f(x+h) - f(x)]}{h} + \lim_{h\to 0} f(x+h) \\
&= x\left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right] + f(x) \\
&= xf'(x) + f(x)
\end{aligned}$$